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Intersecting brane geometries

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Abstract

We present a survey of the calibrated geometries arising in the study of the local singularity structure of supersymmetric fivebranes in M-theory. We pay particular attention to the geometries of 4-planes in eight dimensions, for which we present some new results as well as many details of the computations. We also analyse the possible generalised self-dualities which these geometries can afford.

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1. Introduction

Recent developments suggest that the natural language in which to phrase the study of intersecting branes is that of calibrated geometry. This has proven effective both in the description of their singularity structure [2–4,23] as in the study of the geometry of branes within branes [22]. In our recent work [2–4] on the singularity structure of intersecting branes, which continues the approach laid out in [37,42], we encountered a number of calibrated geometries, not all of which are well known. We believe that it could be useful to present a detailed survey of these geometries in the present context. This is one of the purposes of the present paper. Manifolds admitting special geometries of the kind described here also admit generalisations of the notion of self-duality. In the context of gauge theory, these generalised self-duality give rise to higher-dimensional generalisations of the notion of instanton. We think it useful also to work out the possible generalisations of self-duality that these geometries give rise to.

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This paper is organised as follows. We start in Section 2 by explaining what is meant by a *geometry* in this context and how to translate between its local and global (for lack of a better name) descriptions. Then in Section 3 we will explain how a configuration of intersecting branes gives rise to a geometry, and present a list of all the known geometries which arise in this way. Section 4 is devoted to a case-by-case description of each of these geometries. Section 5 presents a detailed "guided tour" of the different eight-dimensional geometries. The calculational details presented in this section might be of some help to people working in this topic. In Section 6 we explore the different generalisations of self-duality which these calibrated geometries afford. For some of the less common geometries, these results are new. Finally in Section 7 we make some concluding remarks.

2. What is meant by a geometry?

Generally speaking, a geometry is some sort of structure we endow a manifold with. Traditionally a geometry is specified through the existence of certain tensor fields on the manifold. Many well-known geometries arise in this way. For example, a metric gives rise to Riemannian geometry, a closed non-degenerate 2-form to symplectic geometry, and a complex structure gives rise to complex geometry. We can add more structure to one or more of these and obtain many of the geometries which have taken centre stage in recent times: Kähler, hyperkähler, quaternionic Kähler, etc. This way of defining a geometry often goes hand-in-hand with a reduction of the structure group of the frame bundle. On a differential n-manifold without extra structure, the frame bundle is a principal $\mathrm{GL}_n\mathbb{R}$ bundle. With the introduction of a metric, we can consistently restrict ourselves to orthonormal frames, and in effect reduce the structure group to O_n. Similarly, if $n = 2m$, a complex structure allows us to consider complex frames so that the structure group reduces to $\mathrm{GL}_m\mathbb{C}$. More generally, a G-structure on a manifold is a reduction of structure group of the frame bundle to $G \subset \mathrm{GL}_n\mathbb{R}$, so that one can choose local frames which are G-related. Not every manifold admits any G-structure: there might be topological obstructions. For example, although every manifold admits a metric and hence an O_n structure, unless the manifold is orientable it will not admit an SO_n structure.

A common way to reduce the structure group to $G \subset \mathrm{O}_n$ is via a metric whose holonomy lies in G; although not all $G \subset \mathrm{O}_n$ can be so realised, unless one simultaneously allows for torsion in the metric connection. Given a Riemannian manifold M whose metric has holonomy G, the holonomy principle [10] guarantees the existence of privileged tensors on M corresponding to those (algebraic) tensors which are G-invariant. For example, this lies at the heart of the equivalence between the two common definitions of a Kähler manifold: as a Riemannian $2m$-dimensional manifold with U_m holonomy, or as a Riemannian manifold with a parallel complex structure — the complex structure and the metric both being U_m-invariant tensors. This way of specifying a geometry has played an important role in superstring theory, via the study of supersymmetric sigma models particularly.

More recently, however, with the advent of branes, another way of specifying a geometry has become increasingly relevant. Instead of using the existence of tensorial objects or of reductions of the structure group, one specifies a geometry on a manifold by singling out a

special class of submanifolds. For example, one can talk about minimal submanifolds of a Riemannian manifold, or about complex submanifolds of a complex manifold. In fact, in a Kähler manifold these two kinds of submanifolds are not unrelated: complex submanifolds are minimal. The theory of calibrations provides a systematic approach to understanding this fact and allows, in addition, for far-reaching generalisations of this statement. In order to facilitate the discussion it will be necessary to first introduce a few important concepts.

2.1. Submanifolds and Grassmannians

Let N be a p-dimensional submanifold of an n-dimensional manifold M. At each point $x \in N$, the tangent space T_xN to N is a p-dimensional subspace of the tangent space T_xM. In a small enough neighbourhood U of x, we can trivialise the tangent bundle of M. This means essentially that we can identify the tangent space to any point in U with $\mathbb{R}^n$. Now consider those points y in U which also lie in N. Under this identification, the tangent space T_yN of N at y will be identified with a p-plane in $\mathbb{R}^n$. This defines a map from $N \cap U$ to the space of p-planes in $\mathbb{R}^n$. Spaces of planes are generically called *Grassmannians* and will play a central role in the following discussion, so it pays to take a brief look at them before going further.

It is convenient to identify p-planes with certain types of p-vectors. The identification runs as follows. Let π be a p-plane in $\mathbb{R}^n$. Let $e_1, e_2, \ldots, e_p$ be a basis for π. Then the p-vector $e_1 \wedge e_2 \wedge \cdots \wedge e_p$ is non-zero. However, if we choose a different basis $e'_1, e'_2, \ldots, e'_p$ for π, then we generally end up with a different p-vector $e'_1 \wedge e'_2 \wedge \cdots \wedge e'_p$. Of course, both p-vectors are proportional to each other, the constant of proportionality being the (non-zero) determinant of the linear transformation which takes one basis to the other. Conversely, given a non-zero p-vector $v_1 \wedge v_2 \wedge \cdots \wedge v_p$, we associate with it the p-plane π spanned by the $\{v_i\}$, with the proviso that as above, proportional p-vectors give rise to the same p-plane. We can eliminate the multiplicative ambiguity by picking a privileged p-vector for each plane. This can be done by introducing a metric in $\mathbb{R}^n$ and considering only oriented planes. We will reflect this fact by saying that we consider oriented p-planes in the Euclidean space $\mathbb{E}^n$.

Let $G(p|n)$ denote the Grassmannian of oriented p-planes in $\mathbb{E}^n$. As we now show it can be identified with a subspace of the unit sphere in $\mathbb{E}^{\binom{n}{p}}$. Indeed, given an oriented p-plane π, let $e_1, e_2, \ldots, e_p$ be an oriented orthonormal basis and consider the p-vector $e_1 \wedge e_2 \wedge \cdots \wedge e_p \in \bigwedge^p \mathbb{E}^n$, which we will also denote π consistently with the identification we are describing. The norm of any p-vector $v_1 \wedge v_2 \wedge \cdots \wedge v_p$ is given by

$$\|v_1 \wedge v_2 \wedge \cdots \wedge v_p\| = |\det\langle v_i, v_j\rangle|.$$

This norm extends to a metric on $\bigwedge^p \mathbb{E}^n$, which turns it into a Euclidean space $\mathbb{E}^{\binom{n}{p}}$. It follows that π has unit norm, so that it belongs to the unit sphere in $\mathbb{E}^{\binom{n}{p}}$. Conversely, every simple (i.e., decomposable into a wedge product of p vectors) unit p-vector $\pi = e_1 \wedge e_2 \wedge \cdots \wedge e_p \in \bigwedge^p \mathbb{E}^n$ defines an oriented p-plane with basis $e_1, e_2, \ldots, e_p$. In other words, $G(p|n)$ can be identified with a subset of the unit sphere in $\mathbb{E}^{\binom{n}{p}}$, so that it is a compact space. This can also be understood from the fact that the Grassmannian $G(p|n)$

is acted on transitively by SO_n. Indeed it is not hard to see that the isotropy consists of changes of basis in π and its perpendicular $(n-p)$-plane $\pi^\perp$; whence it is isomorphic to $\mathrm{SO}_p \times \mathrm{SO}_{n-p}$. This means that the Grassmannian is a coset manifold:

$$G(p|n) \cong \frac{\mathrm{SO}_n}{\mathrm{SO}_p \times \mathrm{SO}_{n-p}} \cong G(n-p|n).$$

2.2. Geometries and Grassmannians

As mentioned above, a geometry can be specified by singling out a class of special submanifolds. For example, one could consider submanifolds whose tangent spaces belong to a certain subset of the Grassmannian of planes. These give rise to the so-called *Grassmannian geometries*. A special type of subset of the Grassmannian $G(p|n)$ are those sets which correspond to the orbit of a plane under a subgroup of SO_n. It will turn out that all the geometries that we will encounter will be of this form.

For example, suppose that $n = 2m$. Then we could consider complex k-dimensional submanifolds; i.e., $p = 2k$. The tangent subspaces to these submanifolds are k-dimensional complex subspaces of $\mathbb{C}^m \cong \mathbb{R}^n$. All the tangent planes belong to the $\mathrm{U}_m \subset \mathrm{SO}_n$ orbit of any one of the planes. The resulting orbit is the complex Grassmannian $G_{\mathbb{C}}(k|m)$ of k-dimensional complex planes in $\mathbb{C}^m$. It is not hard to see that

$$G_{\mathbb{C}}(k|m) \cong \frac{\mathrm{U}_m}{\mathrm{U}_k \times \mathrm{U}_{m-k}} \cong \frac{\mathrm{SU}_m}{\mathrm{S}(\mathrm{U}_k \times \mathrm{U}_{m-k})},$$

so that, in fact, the planes belong to the same SU_m orbit.

Similarly if $n = 4\ell$, we can consider quaternionic subspaces of $\mathbb{H}^\ell \cong \mathbb{R}^n$. They necessarily have dimension $p = 4j$. The Grassmannian $G_{\mathbb{H}}(j|\ell)$ of quaternionic planes correspond to the orbit of a plane under $\mathrm{Sp}_\ell \subset \mathrm{SO}_n$, so that

$$G_{\mathbb{H}}(j|\ell) \cong \frac{\mathrm{Sp}_\ell}{\mathrm{Sp}_j \times \mathrm{Sp}_{\ell-j}}.$$

Other examples are possible, and we shall discuss them below. For now let us simply point out the fact that for the complex and quaternionic Grassmannians, the subgroups SU_m and Sp_ℓ of SO_n are such that they (or their lifts to subgroups of Spin_n) leave some spinors invariant. This is intimately linked to supersymmetry and will also be the case for the other examples we will encounter. We now turn our attention to another way to single out subsets of the Grassmannian.

2.3. Calibrations

Calibrations will provide us a tool with which to specify subsets (faces, actually) of the Grassmannian of planes. The geometries which are obtained in this fashion are known as *calibrated geometries*. The foundations of this subject are clearly explained in [26], and a shorter but lucid exposition can be found in [35].

Let $\varphi \in \bigwedge^p(\mathbb{E}^n)^*$ be a (constant coefficient) p-form on $\mathbb{E}^n$. It defines a linear function on $\bigwedge^p \mathbb{E}^n$, which restricts to a continuous function on the Grassmannian $G(p|n)$. Because $G(p|n)$ is compact, this function attains a maximum, called the *comass* of φ and denoted $\|\varphi\|^*$. If φ is normalised so that it has comass 1, then it is called a *calibration*. Let $G(\varphi)$ denote those points in $G(p|n)$ on which φ attains its maximum. $G(\varphi)$ is known as the *φ-Grassmannian*, and planes $\pi \in G(\varphi)$ are said to be *calibrated by* φ. The subset $\cup_\varphi G(\varphi) \subset G(p|n)$, where the union runs over all calibrations φ, defines the *faces of* $G(p|n)$. The name comes from the fact that if we think of $G(p|n)$ as a subset of the vector space $\mathbb{E}^{\binom{n}{p}}$, then $G(\varphi)$ is the contact set of $G(p|n)$ with the hyperplane $\{\xi \in \mathbb{E}^{\binom{n}{p}} | \varphi(\xi) = 1\}$. Now, because φ is a calibration, $\varphi(\xi) \leq 1$ and hence $G(p|n)$ lies to one side of that hyperplane.

Computing the comass of a p-form is a difficult problem which has not been solved but for the simplest of forms φ, those which have a high degree of symmetry or those which can be obtained by squaring spinors. Determining the faces of the Grassmannian has proven equally difficult and has only been achieved completely in the lowest dimensions. The determination of the faces of the Grassmannian $G(p|n)$ is not an easy problem whenever p is different from 1, 2, $n-2$, or $n-1$. To this day, only the cases $n = 6$ [14,27,34] and $n = 7$ [28,34] have been fully solved, whereas there are some partial results for $n = 8$ [16]. In the study of static fivebranes in M-theory it is the case $n = 10$ that is needed.

A p-submanifold N of $\mathbb{E}^n$, all of whose tangent planes belong to $G(\varphi)$ for a fixed calibration φ, is said to be a *calibrated submanifold*. A calibrated submanifold N has minimum volume among the set of all submanifolds N' with the same boundary. This is because

$$\operatorname{vol} N = \int_N \varphi = \int_{N'} \varphi \leq \operatorname{vol} N',$$

where the second equality follows by Stokes' theorem. Calibrated submanifolds constitute a far-reaching generalisation of the notion of a geodesic. Indeed, the Grassmannian of oriented lines $G(1|n)$ is just the unit sphere $S^{n-1} \subset \mathbb{E}^n$, whose faces are obviously points. Hence the tangent spaces of a one-dimensional submanifold L belong to the same face if and only if L is a straight line. Notice that there is a duality between p-dimensional and p-codimensional submanifolds; in fact, if φ is a calibration so is $\star\varphi$. Hence hyperplanes in $\mathbb{E}^n$ are also (locally) volume-minimising.

This theory is not restricted to constant coefficient calibrations in $\mathbb{E}^n$. In fact, we can work with d-closed forms φ in any Riemannian manifold (M, g). The comass of φ is now the supremum (over the points in M) of the comasses at each point. If M is compact, this supremum exists. A calibration is now a d-closed form normalised to have unit comass; or equivalent one which satisfies

$$\varphi_x(\xi) \leq \operatorname{vol} \xi \quad \text{for all oriented tangent } p\text{-planes } \xi \text{ at } x.$$

Notice that there may be points in M for which the φ-Grassmannian is empty. The same argument as before shows that calibrated submanifolds are homologically volume-minimising. Of course, this crucially necessitates that φ be d-closed.

If an oriented Riemannian n-manifold has reduced holonomy, meaning a proper subgroup G of SO_n, then the holonomy principle guarantees the existence of parallel (hence d-closed) forms corresponding to the G-invariants in the exterior power of the tangent representation. It turns out that in many (if not all) cases, the parallel forms are calibrations giving rise to interesting geometries. The φ-Grassmannian associated to a G-invariant form φ contains, and in many cases coincides with, the G-orbit of any one of its planes.

3. The (local) geometry of intersecting branes

In this section we summarise the results of [2–4] and tabulate the different geometries that were found. These geometries will be described in more detail in Section 4.

3.1. From branes to geometry

Branes can be understood as certain types of solutions to the supergravity equations of motion. These solutions are characterised by their invariance (at least locally) under a $(p+1)$-dimensional super-Poincaré subalgebra. The solutions describe the exterior spacetime to the worldvolume of a p-dimensional extended object: the brane. The brane therefore corresponds to a $(p+1)$-Lorentzian submanifold, with possible self-intersections. In many cases these submanifolds are minimal and just as for minimal immersions [29,30,33,36] one can ask what is the local singularity structure of a brane solution.

For definiteness we will only discuss fivebranes in 11-dimensional supergravity in this note. It is clear that this approach generalises to general p-branes in this and other supergravities; although it may be possible to treat more general cases from this one by using duality transformations.

Let B be the worldvolume of a fivebrane in an 11-dimensional spin manifold M. Fix a point $x \in B$. Choosing an orthonormal frame [1] $e_0, e_1, \ldots, e_9, e_\natural$ for the tangent space T_xM to M at x, we can identify T_xM with 11-dimensional Minkowski spacetime $\mathbb{E}^{10,1}$. The tangent spaces (if x is a singular point of the immersion then there is more than one) to the worldvolume of a fivebrane passing through x define a subset of the Grassmannian $G(5,1|10,1)$ of time-oriented (5, 1)-planes in $\mathbb{E}^{10,1}$, which analogously to the Euclidean case, is a coset space

$$G(5,1|10,1) \cong \frac{\mathrm{SO}^{\uparrow}_{10,1}}{\mathrm{SO}^{\uparrow}_{5,1} \times \mathrm{SO}_5},$$

where $\mathrm{SO}^{\uparrow}$ stands for the connected component of the identity. The requirement of supersymmetry constraints which subsets of this Grassmannian can the tangents to the branes belong to.

[1] Following [42] we employ the symbol $\natural$ (pronounced 'ten') to refer to the tenth spatial coordinate.

3.2. Supersymmetry

Let $\mathrm{Cl}_{1,10}$ be the Clifford algebra associated to $\mathbb{E}^{10,1}$, but with the opposite norm. In other words, if $v \in \mathbb{E}^{10,1}$ then its Clifford square in $\mathrm{Cl}_{1,10}$ is given by

$$v \cdot v = +\|v\|^2 \mathbb{1},$$

where $\|v\|^2 \equiv -(v^0)^2 + (v^1)^2 + \cdots + (v^{\natural})^2$. As associative algebras $\mathrm{Cl}_{1,10} \cong \mathrm{Mat}_{32}(\mathbb{R}) \oplus \mathrm{Mat}_{32}(\mathbb{R})$, whence it has two inequivalent irreducible representations, each real and 32-dimensional. They are distinguished by the action of the volume element, which takes the values ± 1. Fix one of these irreducible representations Δ once and for all — the choice is immaterial because they are both equivalent under $\mathrm{Spin}_{10,1} \subset \mathrm{Cl}_{1,10}$. Every (5, 1)-plane π in $\mathbb{E}^{10,1}$ defines a subspace

$$\Delta(\pi) \equiv \{\psi \in \Delta | \pi \cdot \psi = \psi\},$$

where $\cdot$ stands for Clifford action and where we have used implicitly the isomorphism of the Clifford algebra $\mathrm{Cl}_{1,10}$ with the exterior algebra. The subspace $\Delta(\pi)$ is non-zero. In fact, because π has unit norm, so that $\pi \cdot \pi = \mathbb{1}$, and zero trace, $\Delta(\pi) \subset \Delta$ is 16-dimensional.

If $\pi_1 \equiv \pi, \pi_2, \ldots, \pi_m$ are m (5, 1)-planes, then we say that the configuration $\cup_{i=1}^m \pi_i$ is *supersymmetric* if and only if

$$\Delta\left(\bigcup_{i=1}^m \pi_i\right) \equiv \bigcap_{i=1}^m \Delta(\pi_i) \neq \{0\}.$$

Moreover, such a supersymmetric configuration is said to preserve a fraction ν of the supersymmetry, whenever

$$32\nu = \dim \Delta\left(\bigcup_{i=1}^m \pi_i\right).$$

A priori ν can only take the values $\frac{1}{32}, \frac{1}{16}, \frac{3}{32}, \ldots, \frac{1}{2}$; although only the following fractions are known to occur: $\frac{1}{32}, \frac{1}{16}, \frac{3}{32}, \frac{1}{8}, \frac{5}{32}, \frac{3}{16}, \frac{1}{4}$ and $\frac{1}{2}$. From the full solution [4,37] of the two fivebrane problem it follows that there are no configurations with fraction $\frac{1}{4} < \nu < \frac{1}{2}$. Therefore the only possible fraction which has yet to appear is $\frac{7}{32}$.

A brane B such that its tangents define a supersymmetric configuration is called a *supersymmetric brane*. An important problem in this topic is the classification of the possible supersymmetric configurations of the so-called intersecting branes (see [21] for a recent review and guide to the literature). Each such configuration gives rise to a subset of the Grassmannian and, by the discussion in Section 2, to a geometry which, as we will see, turns out to be calibrated. This follows from the correspondence between spinors and calibrations, to which we now turn.

3.3. Calibrations and spinors

The relationship between spinors and calibrations is well documented. Although computing the comass of a form φ is generally a difficult problem, it simplifies tremendously when φ can be constructed by squaring spinors. The cleaner results are in seven and eight dimensions [25,31] and more generally in $8k$ dimensions [15]; but similar results can also be obtained in 11 dimensions with Lorentzian signature [3]. Remarkably, it is the 11-and eight-dimensional cases which arise in the study of intersecting branes [3,4].

3.3.1. Eight dimensions

Let us first discuss the eight-dimensional case. As an associative algebra, the Clifford algebra Cl_8 is isomorphic to the matrix algebra $\mathrm{Mat}_{16}(\mathbb{R})$. This means that it has a unique irreducible representation Δ which is real and has dimension 16. Under the spin group $\mathrm{Spin}_8 \subset \mathrm{Cl}_8$, Δ breaks up as $\Delta_+ \oplus \Delta_-$, where each $\Delta_\pm$ corresponds to spinors of definite chirality. Let $\psi \in \Delta_+$ be a chiral spinor, and consider the bispinor $\psi \otimes \bar{\psi}$. It is an element of Cl_8 which, normalising the spinor appropriately, can be written as

$$\psi \otimes \bar{\psi} = \mathbb{1} + \Omega + \mathrm{vol}, \tag{1}$$

where Ω is a self-dual 4-form in $\mathbb{E}^8$. Now let ξ be a simple unit 4-vector in $\mathbb{E}^8$. Then it follows from the expression of the bispinor that $\Omega(\xi)\|\psi\|^2 = \langle\psi, \xi\cdot\psi\rangle$, where $\|\psi\|^2 = \langle\psi,\psi\rangle$ is the norm relative to the natural Spin_8-invariant inner product on Δ_+. By the Cauchy–Schwarz inequality, it follows that

$$\Omega(\xi) = \frac{\langle\psi, \xi\cdot\psi\rangle}{\|\psi\|^2} \leq \frac{\|\xi\cdot\psi\|}{\|\psi\|}.$$

Because ξ belongs to $\mathrm{Spin}_8 \subset \mathrm{Cl}_8$, $\|\xi\cdot\psi\| = \|\psi\|$, whence $\Omega(\xi) \leq 1$ for all ξ. In other words, Ω has unit comass; i.e., it is a calibration. It follows from this argument that the plane defined by the 4-vector ξ is calibrated by Ω if and only if $\xi\cdot\psi = \psi$.

What can one say about the Ω-Grassmannian? The isotropy of a chiral spinor $\psi \in \Delta_+$ is a certain Spin_7^+ subgroup of Spin_8, under which both Δ_- and the vector representation of Spin_8 remain irreducible. This means that Ω is also Spin_7^+-invariant, whence the Spin_7^+-orbit of any plane ξ in the Ω-Grassmannian will also belong to the Ω-Grassmannian. In fact, it is not difficult to show that the Spin_7^+-orbit is the Ω-Grassmannian, which in turn coincides with the Grassmannian of Cayley planes. We will have more to say about this below.

3.3.2. Eleven dimensions

Now let Δ denote one of the two irreducible representations of $\mathrm{Cl}_{1,10}$, and let $\psi \in \Delta$ be a spinor. Squaring the spinor we obtain on the right-hand side a 1-form Ξ, a 2-form Ψ and a 5-form Φ:

$$\psi \otimes \bar{\psi} = \Xi + \Psi + \Phi, \tag{2}$$

where by $\bar{\psi} \equiv -(e_0 \cdot \psi)^{\mathrm{t}}$ we mean the Majorana conjugate. In this expression, the forms Ξ, Ψ and Φ are a 1-, 2-and 5-form in $\mathbb{E}^{10,1}$, respectively. Under the orthogonal decomposition $\mathbb{E}^{10,1} = \mathbb{E}^{10} \oplus \mathbb{R}e_0$, the 5-form Φ breaks up as

$$\Phi = e_0^* \wedge \Lambda + \Theta, \tag{3}$$

where Λ and Θ are a 4-and a 5-form on $\mathbb{E}^{10}$, respectively. Now let ξ be an oriented 5-plane in $\mathbb{E}^{10}$ and consider the bilinear $\bar{\psi}\xi \cdot \psi$. Using (3) and the definition of the Majorana conjugate, one can rewrite this as

$$\langle \psi, (e_0 \wedge \xi) \cdot \psi \rangle = \Theta(\xi)\mathrm{Tr}\,\mathbb{1} = 32\,\Theta(\xi),$$

where we have introduced the Spin_{10}-invariant inner product $\langle -, - \rangle$ defined by $\langle \chi, \psi \rangle = \chi^{\mathrm{t}}\psi$. By the Cauchy–Schwarz inequality for this inner product, we find that

$$\Theta(\xi) \leq \tfrac{1}{32}\|\psi\|\,\|(e_0 \wedge \xi) \cdot \psi\|. \tag{4}$$

Because ξ is a unit simple 5-vector, $\|(e_0 \wedge \xi) \cdot \psi\| = \|\psi\|$, whence

$$\Theta(\xi) \leq \tfrac{1}{32}\|\psi\|^2.$$

In other words, the comass of Θ is given by $\frac{1}{32}\|\psi\|^2$, and a 5-plane ξ is calibrated by Θ if and only if the (5, 1) plane $\pi = e_0 \wedge \xi$ obeys $\pi \cdot \psi = \psi$, which is precisely the condition that ψ belongs to $\Delta(\pi)$.

The nature of the Θ-Grassmannian depends on the isotropy group of the spinor ψ. A non-zero Majorana spinor ψ of $\mathrm{Spin}_{10,1}$ can have two possible isotropy groups [11]: either $\mathrm{SU}_5 \subset \mathrm{Spin}_{10}$, which acts trivially on a time-like direction which can be chosen to be e_0, or a 30-dimensional non-semisimple Lie group $G \cong \mathrm{Spin}_7 \ltimes \mathbb{R}^9$, acting trivially on a null direction. In the former case, the 5-form Θ is SU_5-invariant and the Θ-Grassmannian will contain the SU_5-orbit of the plane π. This orbit turns out to be the full Θ-Grassmannian, which is the Grassmannian of special Lagrangian planes in $\mathbb{E}^{10}$. In the latter case, Θ has the form $v^* \wedge \Omega$ where Ω is a Cayley calibration in an eight-dimensional subspace $V \subset \mathbb{E}^{10}$ and $v \in V^{\perp}$ is a fixed vector perpendicular to V. In this case the Θ-Grassmannian agrees with the Ω-Grassmannian, which is isomorphic to the Grassmannian of Cayley planes in $V \cong \mathbb{E}^8$.

3.4. Summary of results

We can summarise the foregoing discussion as follows. Given any supersymmetric configuration of M5-branes, the tangent planes $\{\pi_i\}$ at any given singular point belong to a face of the Grassmannian: the intersection of the faces corresponding to all the spinors ψ which belong to $\Delta(\pi_i)$ for all i. We will call such a face of the Grassmannian, a *supersymmetric face*. The main problem in the study of the local singularity structure of supersymmetric M5-branes is the determination of the supersymmetric faces of the Grassmannian $G(5, 1|10, 1)$ of (5, 1)-planes in $\mathbb{E}^{10,1}$, and for each such face to determine the fraction ν of the supersymmetry which is preserved.

The first attempt at solving this problem was by Ohta and Townsend [37] who, following up the work in [42], classified the supersymmetric static configurations of a pair of M5-branes. (Some earlier incomplete results can be found in [40].) The solution of the two fivebrane problem was completed in [4], where we also considered fivebranes which are moving relative to each other. The multiple brane problem is still open, but some partial results can be found in [2–4,23]. As explained in [2,4], but see also [3,8,23], the supersymmetric faces consist of planes which lie in the orbit of one of the planes under the action of a subgroup of $\mathrm{Spin}_{10,1}$ which leaves invariant some subspace of Δ. For each such subgroup G one can determine the fraction ν of the supersymmetry which is preserved and the geometry defined by its orbit in the face of the Grassmannian.

We can distinguish two cases: faces in which all planes share a common time-like direction and faces in which all planes share a common light-like direction. The former correspond to static brane configurations, whereas the latter correspond to branes in motion. Moreover, as shown in [4], supersymmetric configurations of branes are obtained by null-rotating (see, e.g., [39]) already supersymmetric configurations consisting of Cayley planes in eight dimensions.

We summarise the known results in Table 1. Each of the geometries in the table is defined as the G-orbit of a p-plane in $\mathbb{E}^n$. For each such geometry we also list the isotropy subgroup $H \subset G$ of the reference p-plane, as well as the type of (calibrated) geometry which one

Table 1
Some of the geometries associated with intersecting brane configurations, together with the fraction of the supersymmetry which is preserved both for static and for moving branes

$(p\|n)$	Group G	Isotropy H	Geometry G/H	Fraction ν	
				Static	Moving
(5|10)	SU_5	SO_5	SLAG_5	$\frac{1}{32}$	–
(5|10)	$\mathrm{SU}_2 \times \mathrm{SU}_3$	$\mathrm{SO}_2 \times \mathrm{SO}_3$	$\mathrm{SLAG}_2 \times \mathrm{SLAG}_3$	$\frac{1}{16}$	–
(4|8)	Spin_7	$(\mathrm{SU}_2)^3/\mathbb{Z}_2$	Cayley	$\frac{1}{32}$	$\frac{1}{32}$
(4|8)	SU_4	SO_4	SLAG_4	$\frac{1}{16}$	$\frac{1}{32}$
(4|8)	SU_4	$\mathrm{S}(\mathrm{U}_2 \times \mathrm{U}_2)$	$G_{\mathbb{C}}(2\|4)$	$\frac{1}{16}$	$\frac{1}{16}$
(4|8)	Sp_2	U_2	$\mathbb{C}\mathrm{LAG}_2$	$\frac{3}{32}$	$\frac{1}{32}, \frac{1}{16}$
(4|8)	Sp_2	$\mathrm{Sp}_1 \times \mathrm{Sp}_1$	$G_{\mathbb{H}}(1\|2)$	$\frac{3}{32}$	$\frac{3}{32}$
(4|8)	$\mathrm{Sp}_1 \times \mathrm{Sp}_1$	$\mathrm{U}_1 \times \mathrm{U}_1$	$(G_{\mathbb{C}}(1\|2))^2$	$\frac{1}{8}$	$\frac{1}{16}$
(4|8)	$\mathrm{Sp}_1 \times \mathrm{Sp}_1$	Sp_1	(3, 1)	$\frac{1}{8}$	$\frac{1}{32}, \frac{3}{32}$
(4|8)	Sp_1	U_1	(3, 2)	$\frac{5}{32}$	$\frac{1}{16}, \frac{3}{32}$
(4|8)	U_1	{1}	(3, 3)	$\frac{3}{16}$	$\frac{3}{32}$
(3|7)	G_2	SO_4	Associative	$\frac{1}{16}$	$\frac{1}{16}$
(3|6)	SU_3	SO_3	SLAG_3	$\frac{1}{8}$	$\frac{1}{16}$
(2|6)	SU_3	$\mathrm{S}(\mathrm{U}_2 \times \mathrm{U}_1)$	$G_{\mathbb{C}}(1\|3)$	$\frac{1}{8}$	$\frac{1}{8}$
(2|4)	SU_2	SO_2	SLAG_2	$\frac{1}{4}$	$\frac{1}{8}$

obtains. We also tabulate the fractions of supersymmetry both for static and (when $n \leq 8$) moving branes. Some entries have more than one possible fraction ν for moving branes. These correspond to different but isomorphic subgroups G. The static fraction depends only on the conjugacy class of G in Spin_{10}, but the moving fraction is a more subtle invariant of the configuration and depends intricately on how G sits in Spin_8.

It may prove useful to explain one of the entries in detail. Let us consider for instance the fourth row in the table. These configurations are obtained as follows. For static configurations, pick a (5, 1)-plane $\pi = e_0 \wedge \xi$, where ξ is a 5-plane in $e_0^\perp \cong \mathbb{E}^{10}$. The allowed configurations consist of planes $\pi' = e_0 \wedge \xi'$, where ξ' is in the orbit of ξ under a subgroup $G \cong \mathrm{SU}_4$. G leaves one direction invariant, v say, in ξ, so that the plane π can be written as $\pi = e_0 \wedge v \wedge \zeta$, where ζ is a 4-plane in the eight-dimensional subspace of $e_0^\perp$ on which SU_4 acts irreducibly. All other planes will be of the form $\pi' = e_0 \wedge v \wedge \zeta'$ where ζ' is in the G-orbit of ζ. The isotropy (in G) of ζ is a subgroup $H \cong \mathrm{SO}_4$ and with a little more effort one can recognise the subset of the Grassmannian as consisting of the special Lagrangian 4-planes. Those configurations will generically preserve $\frac{1}{16}$ of the supersymmetry. For moving branes one simply starts with a configuration of static branes, namely planes of the form $\pi' = e_0 \wedge v \wedge \zeta'$ where ζ' a special Lagrangian 4-plane, and performs an arbitrary null rotation to each of the planes. Only null rotations along directions perpendicular to the plane π' change the configuration, whence the resulting Grassmannian is a homogeneous bundle over G/H with fibre $\mathbb{R}^5$. The generic configuration now preserves $\frac{1}{32}$ of the supersymmetry.

4. Some geometries associated with intersecting branes

We now start a case-by-case description of the geometries in Table 1. These geometries are not new, of course, but some may not be well-known. Complex geometries are of course classical, and to some extent so are quaternionic geometries. The special Lagrangian, associative and Cayley geometries were discussed initially in Harvey & Lawson's [26] foundational essay on calibrated geometry. The complex Lagrangian geometry (at least in dimension eight) as well as the other geometries associated to self-dual 4-forms are discussed in [16].

4.1. Complex geometry

The complex geometry of k-planes in $\mathbb{C}^m \cong \mathbb{R}^{2m}$ is defined by the Grassmannian $G_{\mathbb{C}}(k|m) \subset G(2k|2m)$. It is the $\mathrm{SU}_m \subset \mathrm{SO}_{2m}$ orbit of a given real $2k$-plane. Such planes are calibrated by the properly normalised kth power of the Kähler form

$$\omega = \sum_{i=1}^{m} \mathrm{d}x^i \wedge \mathrm{d}y^i,$$

where $z^i = x^i + \sqrt{-1}y^i$ are complex coordinates. It follows from Wirtinger's inequality (see, e.g., [18]) that the $2k$-form $(1/k!)\omega^k$ has unit comass and that its Grassmannian is

precisely the Grassmannian of complex k-planes. The Kähler form is left invariant by a U_m subgroup of SO_{2m}, whose intersection with the isotropy $\mathrm{SO}_{2k} \times \mathrm{SO}_{2m-2k}$ (in SO_{2m}) of a real plane is $\mathrm{U}_k \times \mathrm{U}_{m-k}$. Note however that the centre of U_m acts trivially, whence factoring it out, we can write

$$G_{\mathbb{C}}(k|m) \cong \frac{\mathrm{SU}_m}{\mathrm{S}(\mathrm{U}_k \times \mathrm{U}_{m-k})}.$$

In the study of M5-branes we have the following Grassmannians appearing: $G_{\mathbb{C}}(1|3)$, $G_{\mathbb{C}}(2|4) \cong G(2|6)$, and $G_{\mathbb{C}}(1|2) \times G_{\mathbb{C}}(1|2) \cong G(2|4)$.

4.2. Quaternionic geometry

Consider $\mathbb{H}^\ell \cong \mathbb{R}^{4\ell}$ and, on it, the quaternionic 4-form

$$\Theta = \sum_{i=1}^{\ell} \mathrm{d}x^i \wedge \mathrm{d}y^i \wedge \mathrm{d}z^i \wedge \mathrm{d}w^i,$$

where a quaternionic vector has components $q^i = x^i\mathbb{i} + y^i\mathbb{j} + z^i\mathbb{k} + w^i\mathbb{1}$. Then results of Berger [7] show that the $4j$-form $(6/(2j+1)!)\Theta^j$ has unit comass, and the corresponding Grassmannian is nothing but the Grassmannian $G_{\mathbb{H}}(j|\ell) \subset G(4j|4\ell)$ of quaternionic j-planes. This Grassmannian is acted on transitively by $\mathrm{Sp}_\ell \subset \mathrm{SO}_{4\ell}$, and the intersection of Sp_ℓ with the isotropy $\mathrm{SO}_{4j} \times \mathrm{SO}_{4\ell-4j}$ of a real $4j$-plane, is given by $\mathrm{Sp}_j \times \mathrm{Sp}_{\ell-j}$, whence

$$G_{\mathbb{H}}(j|\ell) \cong \frac{\mathrm{Sp}_\ell}{\mathrm{Sp}_j \times \mathrm{Sp}_{\ell-j}}.$$

In the above table, it is $G_{\mathbb{H}}(1|2) \cong G(1|5) \cong S^4$ which appears.

4.3. Special Lagrangian geometry

Special Lagrangian geometry is another geometry associated to $\mathrm{SU}_m \subset \mathrm{SO}_{2m}$. This geometry is complementary to the geometry of complex planes in $\mathbb{C}^m \cong \mathbb{R}^{2m}$. Indeed, it is a geometry of totally real planes. Consider the forms

$$\Lambda^{(\theta)} = \mathrm{Re}\,\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \cdots \wedge \mathrm{d}z^m,$$

where z^i are the complex coordinates for $\mathbb{C}^m$ introduced in Section 4.1 and $\theta \in S^1$. It is shown in [26] that $\Lambda^{(\theta)}$ has unit comass, so that it is a calibration. Its Grassmannian consists of the so-called *special Lagrangian* planes. These planes are Lagrangian with respect to the Kähler form ω on $\mathbb{C}^m$ defined in Section 4.1: i.e., they are maximally isotropic relative to ω. Notice however that the subset of $G(m|2m)$ consisting of *all* Lagrangian planes (with respect to ω) is not the φ-Grassmannian for any φ. Nevertheless, it is fibred over the circle with fibres the special Lagrangian planes relative to $\Lambda^{(\theta)}$, for $\theta \in S^1$. In other words, every Lagrangian plane is special Lagrangian with respect to $\Lambda^{(\theta)}$ for some θ. Notice that

$\mathrm{U}_m \subset \mathrm{SO}_{2m}$ does *not* preserve $\Lambda^{(\theta)}$ now, since the centre shifts θ; but SU_m does. Its intersection with the isotropy $\mathrm{SO}_m \times \mathrm{SO}_m$ of an m-plane is the diagonal SO_m, whence the special Lagrangian Grassmannian SLAG_m can be written as

$$\mathrm{SLAG}_m \cong \frac{\mathrm{SU}_m}{\mathrm{SO}_m}.$$

Notice that for $m = 2$, special Lagrangian geometries can be identified with complex geometries relative to a complex structure which is also left invariant by the same SU_2 subgroup. This is because $\mathrm{SU}_2 \cong \mathrm{Sp}_1$ actually leaves invariant a quaternionic structure on $\mathbb{H} \cong \mathbb{C}^2$. Other special Lagrangian geometries which appear in the table are SLAG_5, $\mathrm{SLAG}_4 \cong G(3|6)$, and SLAG_3.

4.4. Associative geometry

Associative and Cayley geometries are intimately linked to the octonions. There are many constructions of the calibrations which define these geometries, but they are all in one way or another related to the octonions. Let us therefore consider the 3-form φ on $\mathbb{R}^7$ defined as follows. We identify $\mathbb{R}^7 \cong \mathrm{Im}\,\mathbb{O}$ with the imaginary octonions. The octonions are a normed algebra, whence in addition to a multiplication $\cdot$ they also have an inner product $\langle , \rangle$. The 3-form φ is defined by

$$\varphi(a, b, c) = \langle a, b \cdot c \rangle$$

for all $a, b, c \in \mathrm{Im}\,\mathbb{O}$. We can choose a basis o_i, $i = 1, \ldots, 7$, for the imaginary octonions, and canonically dual basis θ_i, relative to which φ can be written as

$$\varphi = \theta_{125} + \theta_{136} + \theta_{147} - \theta_{237} + \theta_{246} - \theta_{345} + \theta_{567},$$

where we have used the shorthand $\theta_{ijk} \equiv \theta_i \wedge \theta_j \wedge \theta_k$. Harvey and Lawson [26] proved that φ is a calibration. The φ-Grassmannian consists of the so-called *associative* planes, which are all constructed as follows. Let $\mathbb{i}, \mathbb{j}, \mathbb{k}$ generate any quaternion subalgebra of $\mathbb{O}$. Then the 3-plane $\mathbb{i} \wedge \mathbb{j} \wedge \mathbb{k}$ is associative, and moreover all associative planes are constructed in this way.

The group of automorphisms of the octonions is G_2, and its action is such that it stabilises $\mathrm{Im}\,\mathbb{O}$. It also preserves the inner product, whence it leaves φ invariant. In fact, G_2 can be defined [12] as the subgroup of $\mathrm{GL}_7\mathbb{R}$ which leaves φ invariant. The isotropy (in G_2) of an associative plane is isomorphic to an SO_4 subgroup, which acts on $\mathrm{Im}\,\mathbb{O}$ as follows. We identify $\mathrm{Im}\,\mathbb{O}$ with $\mathrm{Im}\,\mathbb{H} \oplus \mathbb{H}$ and SO_4 with the $\mathbb{Z}_2$ quotient of $\mathrm{Sp}_1 \times \mathrm{Sp}_1$, with Sp_1 the unit imaginary quaternions. Thus if $g = (q_1, q_2) \in \mathrm{Sp}_1 \times \mathrm{Sp}_1$, then

$$g(a, b) = (q_1 \cdot a \cdot \bar{q}_1, q_2 \cdot b \cdot \bar{q}_1)$$

for $a \in \mathrm{Im}\,\mathbb{H}$ and $b \in \mathbb{H}$. Notice that the $\mathbb{Z}_2$ subgroup generated by $(-\mathbb{1}, -\mathbb{1}) \in \mathrm{Sp}_1 \times \mathrm{Sp}_1$ acts trivially. Clearly $\mathbb{i} \wedge \mathbb{j} \wedge \mathbb{k}$ is left invariant by SO_4 and it is shown in [26] that SO_4 is

precisely the isotropy of this 3-plane. In summary, the associative Grassmannian is given by

$$\text{Associative} \cong \frac{G_2}{\mathrm{SO}_4}.$$

4.5. *Cayley geometry*

The Cayley Grassmannian is the face exposed by a self-dual 4-form Ω in $\mathbb{R}^8$, which we identify with $\mathbb{O}$ as before. Indeed, we can build Ω in terms of the associative 3-form φ defined above, in the following way. Consider the Hodge dual $\tilde{\varphi}$ (in $\mathrm{Im}\,\mathbb{O}$) of φ:

$$\tilde{\varphi} \equiv \star_7\varphi = \theta_{1234} - \theta_{1267} + \theta_{1357} - \theta_{1456} + \theta_{2356} + \theta_{2457} + \theta_{3467},$$

in the obvious notation. Thinking of $\tilde{\varphi}$ as a 4-form in $\mathbb{O}$, its Hodge dual is given by $\varphi \wedge \theta_8$, where θ_8 is the canonical dual form to $\mathbb{1} \in \mathbb{O}$. We can now define a self-dual 4-form Ω in $\mathbb{O}$ as follows:

$$\begin{aligned}\Omega = \tilde{\varphi} + \varphi \wedge \theta_8 = {} & \theta_{1234} + \theta_{1258} - \theta_{1267} + \theta_{1357} + \theta_{1368} - \theta_{1456} + \theta_{1478} + \theta_{2356} \\ & -\theta_{2378} + \theta_{2457} + \theta_{2468} - \theta_{3458} + \theta_{3467} + \theta_{5678}.\end{aligned}$$

As proven in [26], Ω has unit comass. It is known as the *Cayley calibration*, and its calibrated planes make up the *Cayley Grassmannian*. Alternatively, Ω can be defined in terms of the inner product on $\mathbb{O}$ and the *triple cross product*

$$a \times b \times c = \tfrac{1}{2}(a \cdot (\bar{b} \cdot c) - c \cdot (\bar{b} \cdot a))$$

as follows:

$$\Omega(a, b, c, d) = \langle a \times b \times c, d \rangle.$$

It follows that the typical calibrated plane is of the form $\mathbb{1} \wedge \mathbb{i} \wedge \mathbb{j} \wedge \mathbb{k}$, where $\mathbb{i}$, $\mathbb{j}$, and $\mathbb{k} = \mathbb{i} \cdot \mathbb{j}$ are the imaginary units in a quaternion subalgebra of $\mathbb{O}$.

The Cayley form Ω is invariant under a Spin_7 subgroup of SO_8, which acts transitively on the unit sphere in O with isotropy G_2. As in the associative case, Spin_7 can be defined as the subgroup of $\mathrm{GL}_8\mathbb{R}$ which leaves Ω invariant. It follows that Spin_7 acts on the Cayley Grassmannian. This action is transitive, with isotropy a subgroup $H \cong (\mathrm{Sp}_1 \times \mathrm{Sp}_1 \times \mathrm{Sp}_1)/\mathbb{Z}_2$ which acts on $\mathbb{O}$ as follows. If $g = (q_1, q_2, q_3) \in \mathrm{Sp}_1 \times \mathrm{Sp}_1 \times \mathrm{Sp}_1$ is a triple of unit imaginary quaternions, then under $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}$ we have

$$g(a, b) = (q_3 \cdot a \cdot \bar{q}_1, q_2 \cdot b \cdot \bar{q}_1)$$

for $a, b \in \mathbb{H}$ Notice that $(-1, -1, -1)$ acts trivially, whence the action factors through H. Clearly H leaves $\mathbb{1} \wedge \mathbb{i} \wedge \mathbb{j} \wedge \mathbb{k}$ invariant, and it is shown in [26] that H is precisely the isotropy of such a plane. In summary, the Cayley Grassmannian can be written as

$$\text{Cayley} \cong \frac{\mathrm{Spin}_7}{H}.$$

Notice that the Cayley Grassmannian is isomorphic to $G(3|7)$. This is no accident, since given any oriented 3-plane in $\mathbb{R}^7$, there is a unique Cayley plane in $\mathbb{R}^8$ which contains it.

4.6. Complex Lagrangian geometry

The complex Lagrangian geometry is a geometry of 2ℓ-planes in $\mathbb{R}^{4\ell}$. Identifying $\mathbb{R}^{4\ell}$ with $\mathbb{H}^\ell$, determines a quaternionic structure I, J, and $K = IJ$. The complex Lagrangian planes are those planes which are complex relative to I, say, and Lagrangian relative to J. Let ω_I denote the Kähler form relative to I, and $\Lambda^{(0)}_J$ denote the special Lagrangian form relative to J with angle $\theta = 0$. Then consider the sum

$$\Xi = \tfrac{1}{2}\Lambda^{(0)}_J + \tfrac{1}{2}\tfrac{1}{\ell!}\omega_I^\ell.$$

One can show that Ξ is a calibration, whose Grassmannian consists of those real 2ℓ-planes which are complex relative to I and (special) Lagrangian relative to J; i.e., the complex Lagrangian 2ℓ-planes. The quaternionic structure $\{I, J, K\}$ determines an Sp_ℓ subgroup of $\mathrm{SO}_{4\ell}$, which leaves Ξ invariant. Its intersection with the isotropy of a reference complex Lagrangian plane is a U_ℓ subgroup, whence

$$\mathbb{C}\mathrm{LAG}_\ell \cong \frac{\mathrm{Sp}_\ell}{\mathrm{U}_\ell}.$$

Notice that $\mathbb{C}\mathrm{LAG}_1 \cong \mathrm{SLAG}_2 \cong G_{\mathbb{C}}(1|2)$. Apart from this degenerate case, it is $\mathbb{C}\mathrm{LAG}_2 \cong G(2|5)$ which appears in the table. In this case, it is not hard to show that Ξ is actually self-dual, as was the case for the Cayley, quaternionic and complex geometries of real 4-planes in $\mathbb{R}^8$ discussed above, and for the remaining three calibrations to be discussed below.

4.7. Other geometries associated to self-dual 4-forms

It remains to discuss the three geometries labelled (3, 1), (3, 2) and (3, 3) in the table. The notation has been borrowed from [16] who classified the (anti-)self-dual calibrations in $\mathbb{R}^8$, of which these are examples.

Each one in turn is associated to a certain self-dual calibration on $\mathbb{R}^8$. Let us choose an oriented basis e_i for $\mathbb{R}^8$ and let θ_i denote the canonical dual basis. We will use the notation where $e_{ijk\ell} = e_i \wedge e_j \wedge e_k \wedge e_\ell$ and similarly for $\theta_{ijk\ell}$. In addition let

$$\theta^{ijk\ell} = \theta_{ijk\ell} + \star\theta_{ijk\ell}$$

be the manifestly self-dual extension of $\theta_{ijk\ell}$. Consider the following three self-dual forms

$$\begin{aligned}
\Psi_{(3,1)} &= \theta^{1234} + \tfrac{1}{2}\theta^{1256} + \tfrac{1}{2}\theta^{1467} - \tfrac{1}{2}\theta^{1368},\\
\Psi_{(3,2)} &= \theta^{1234} + \tfrac{3}{5}\theta^{1256} - \tfrac{1}{5}\theta^{1278} + \tfrac{1}{5}\theta^{1357} + \tfrac{1}{5}\theta^{1467} - \tfrac{1}{5}\theta^{1368} + \tfrac{1}{5}\theta^{1458},\\
\Psi_{(3,3)} &= \theta^{1234} + \tfrac{1}{3}\theta^{1256} - \tfrac{1}{3}\theta^{1368} + \tfrac{1}{3}\theta^{1458}.
\end{aligned} \tag{5}$$

As shown in [16] these forms have unit comass. It is clear from their explicit expressions that the 4-plane e_{1234} is calibrated by each of them. These forms are left invariant by the following subgroups of SO_8:

$$K_{(3,1)} = \mathrm{Sp}_1 \cdot (\mathrm{Sp}_1 \times \mathrm{Sp}_1) \cong \frac{(\mathrm{Sp}_1 \times \mathrm{Sp}_1 \times \mathrm{Sp}_1)}{\mathbb{Z}_2},$$
$$K_{(3,2)} = \mathrm{Sp}_1 \cdot (\mathrm{Sp}_1 \times \mathrm{U}_1) \cong \frac{(\mathrm{Sp}_1 \times \mathrm{Sp}_1 \times \mathrm{U}_1)}{\mathbb{Z}_2},$$
$$K_{(3,3)} = \mathrm{Sp}_1 \cdot (\mathrm{U}_1 \times \mathrm{Sp}_1) \cong \frac{(\mathrm{Sp}_1 \times \mathrm{U}_1 \times \mathrm{Sp}_1)}{\mathbb{Z}_2}, \tag{6}$$

which are all subgroups of the $\mathrm{Sp}_1 \cdot \mathrm{Sp}_2 \cong (\mathrm{Sp}_1 \times \mathrm{Sp}_2)/\mathbb{Z}_2$ subgroup which leaves invariant the quaternionic form

$$\Theta = \theta^{1234} + \tfrac{1}{3}\theta^{1256} + \tfrac{1}{3}\theta^{1278} + \tfrac{1}{3}\theta^{1357} - \tfrac{1}{3}\theta^{1368} + \tfrac{1}{3}\theta^{1458} + \tfrac{1}{3}\theta^{1467},$$

which also calibrates e_{1234}. This shows that these geometries are subgeometries of the quaternionic geometry $G(\Theta) \cong G_{\mathbb{H}}(1|2)$. As shown in [16], the Grassmannians $G(\Psi_{(3,i)})$ coincide with the $K_{(3,i)}$ orbits of e_{1234}. Computing the intersection of the $\mathrm{SO}_4 \times \mathrm{SO}_4 \subset \mathrm{SO}_8$ isotropy subgroup of e_{1234} with the $K_{(3,i)}$ and factoring out common subgroups, we obtain the following description for the Grassmannians

$$G(\Psi_{(3,1)}) \cong \frac{\mathrm{Sp}_1 \times \mathrm{Sp}_1}{\mathrm{Sp}_1} \cong G(1|4) \cong S^3,$$
$$G(\Psi_{(3,2)}) \cong \frac{\mathrm{Sp}_1}{\mathrm{U}_1} \cong G(1|3) \cong S^2,$$
$$G(\Psi_{(3,1)}) \cong \mathrm{U}_1 \cong G(1|2) \cong S^1.$$

5. The eight-dimensional geometries in detail

In this section we will go in more detail through the eight-dimensional geometries in Table 1 — i.e., the subgeometries of $G(4|8)$. There are nine such geometries: Cayley, complex (two kinds), quaternionic, special Lagrangian, complex Lagrangian, as well as the $(3, i)$ subgeometries of the quaternionic geometry. All these geometries share the property that they are calibrated by self-dual 4-forms in $\mathbb{R}^8$. The strategy in this section is the following. We fix a given 4-plane in $\mathbb{E}^8$ and we will describe the orbits of this plane under different subgroups of SO_8. In many cases, these subgroups will be determined uniquely by specifying a certain structure (complex, quaternionic, etc.) in $\mathbb{E}^8$ which it leaves invariant.

5.1. Notation and basic strategy

We will let $\{e_i\}$ for $i = 1, 2, \ldots, 8$ be an oriented orthonormal basis for $\mathbb{E}^8$, and introduce the shorthand notation $e_{ij\ldots k} = e_i \wedge e_j \wedge \cdots \wedge e_k$. This choice of basis allows us to identify $\mathbb{E}^8$ with its dual, and forms with polyvectors. Our reference oriented 4-plane will be e_{1234}. Its SO_8-isotropy K is isomorphic to $\mathrm{SO}_4 \times \mathrm{SO}_4$, the first factor acting on the span of e_{1234}

and the second on the span of e_{5678}. The Grassmannian of oriented 4-planes in $\mathbb{E}^8$ is then the SO_8-orbit of e_{1234},

$$G(4|8) = \mathrm{SO}_8 \cdot e_{1234} \cong \frac{\mathrm{SO}_8}{\mathrm{SO}_4 \times \mathrm{SO}_4}.$$

For every subgroup $G \subset \mathrm{SO}_8$, the G-orbit of e_{1234} is a subset of the Grassmannian which is itself isomorphic to a coset space,

$$G(4|8) \supset G \cdot e_{1234} \cong \frac{G}{G \cap K}.$$

In what follows we will specify the group G in terms of invariant structures on $\mathbb{E}^8$.

It is well known that a complex structure determines an SU_4 subgroup of SO_8 which shares its maximal torus with a Spin_7 subgroup. Also a quaternionic structure determines an Sp_2 subgroup of SO_8. This Sp_2 subgroup is nothing but the intersection of the SU_4 subgroups corresponding to each of the three complex structures in the quaternionic structure. A quaternionic structure allows us to think of $\mathbb{E}^8$ as $\mathbb{H}^2$. A given split $\mathbb{H}^2 = \mathbb{H} \oplus \mathbb{H}$ is preserved by an $\mathrm{Sp}_1 \times \mathrm{Sp}_1$ subgroup of the Sp_2, and this $\mathrm{Sp}_1 \times \mathrm{Sp}_1$ subgroup in turn determines a diagonal Sp_1 subgroup, whose maximal torus defines a U_1 subgroup. Starting with different complex structures and some extra structure along the way, we will therefore be able to construct all the geometries of interest.

5.2. A guided tour

We start, following [16], by introducing a convenient notation for complex structures in $\mathbb{E}^8$. By a complex structure

$$I = \begin{pmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{pmatrix}, \tag{7}$$

we mean that $I\, e_1 = e_2$, $I\, e_2 = -e_1$, $I\, e_3 = e_4$, etc. Each complex structure determines a "Kähler" 2-form, which in this case is given by

$$\omega_I = e_{12} + e_{34} + e_{56} + e_{78},$$

which in turn defines a self-dual 4-form, called the Kähler calibration:

$$\tfrac{1}{2}\omega_I^2 = e^{1234} + e^{1256} + e^{1278}, \tag{8}$$

where as above we have introduced the explicit self-dual 4-forms

$$e^{ijkl} = e_{ijkl} + \star e_{ijkl}.$$

A complex structure I also defines a special Lagrangian calibration Λ_I in the following way. We start by defining the following complex vectors:

$$\zeta_i = e_{2i-1} + \sqrt{-1}\, e_{2i}$$

for $i = 1, 2, 3, 4$. They have the virtue that they are eigenvectors of I and therefore "diagonalise" the Kähler form:

$$\omega_I = \tfrac{1}{2}\mathrm{Im}\sum_{i=1}^{4}\zeta_i \wedge \bar{\zeta}_i.$$

The special Lagrangian calibration Λ_I is then defined as the following real 4-form:

$$\Lambda_I = \mathrm{Re}(\zeta_1 \wedge \zeta_2 \wedge \zeta_3 \wedge \zeta_4),$$

expanding to

$$\Lambda_I = e^{1357} - e^{1368} - e^{1467} - e^{1458},$$

which is manifestly self-dual. Notice that Λ_I does not calibrate e_{1234}. This is to be expected because a plane cannot be both complex and Lagrangian (hence totally real) relative to the same complex structure.

Therefore we choose a second complex structure J defined by

$$J = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 8 & 7 & 6 & 5 \end{pmatrix}. \tag{9}$$

Its Kähler form is given by $\omega_J = e_{18} + e_{27} + e_{36} + e_{45}$, which squares to

$$\tfrac{1}{2}\omega_J^2 = e^{1278} + e^{1368} + e^{1458}. \tag{10}$$

The special Lagrangian form is given by

$$\Lambda_J = e^{1234} + e^{1256} - e^{1357} + e^{1467}, \tag{11}$$

which clearly calibrates e_{1234}. Whereas the special Lagrangian calibration Λ_J is SU_4-invariant, the Kähler calibration $\frac{1}{2}\omega_J^2$ is actually U_4-invariant. Nevertheless, the centre of U_4, being generated by the complex structure J itself, stabilises the plane, whence, just as for the special Lagrangian Grassmannian, the complex Grassmannian is an SU_4 orbit.

Now consider the combination

$$\Omega_J = \Lambda_J - \tfrac{1}{2}\omega_J^2.$$

As shown in [26], this is a Cayley form and is left invariant by the Spin_7 subgroup of SO_8 which contains (and shares the same maximal torus with) the SU_4 leaving J invariant. In our case, Ω_J expands to

$$\Omega_J = e^{1234} + e^{1256} - e^{1278} - e^{1357} - e^{1368} - e^{1458} + e^{1467}, \tag{12}$$

from which we see that it calibrates e_{1234}.

The two complex structures I and J defined above anticommute: $K \equiv IJ = -JI$, where

$$K = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -7 & 8 & -5 & 6 \end{pmatrix}, \tag{13}$$

correcting a typo in [16]. Therefore $\{I, J, K\}$ define a quaternionic structure on $\mathbb{E}^8$. The intersection of the SU_4 subgroups corresponding to the three complex structures is an Sp_2 subgroup of SO_8. Given an Sp_2 subgroup it gives rise to a family of 24 quaternionic structures: all possible reorderings and consistent sign changes in $\{I, J, K\}$. The Kähler and special Lagrangian calibrations for each of the complex structures in the quaternionic structure satisfy a number of useful identities:

$$\Lambda_I = \tfrac{1}{2}\omega_K^2 - \tfrac{1}{2}\omega_J^2, \qquad \Lambda_J = \tfrac{1}{2}\omega_I^2 - \tfrac{1}{2}\omega_K^2, \qquad \Lambda_K = \Lambda_I + \Lambda_J. \tag{14}$$

A useful way to construct new calibrations out of old ones is to take *convex* linear combinations. By this we mean a linear combination $\sum_i a_i C_i$, where each C_i is a calibration and $a_i \geq 0$ with $\sum_i a_i = 1$. Such a linear combination is automatically a calibration and moreover its Grassmannian is the intersection of the C_i-Grassmannians. Because e_{1234} belongs to both the complex Grassmannian corresponding to I and to the special Lagrangian Grassmannian corresponding to J, we can take the following convex linear combination:

$$\Xi = \tfrac{1}{4}\omega_I^2 + \tfrac{1}{2}\Lambda_J,$$

which expands to

$$\Xi = e^{1234} + e^{1256} + \tfrac{1}{2}e^{1278} - \tfrac{1}{2}e^{1357} + \tfrac{1}{2}e^{1467}. \tag{15}$$

Its Grassmannian consists of those planes which are complex with respect to I and special Lagrangian with respect to J. The resulting geometry is called complex Lagrangian. The same geometry arises as the calibrated geometry of the convex linear combination

$$\Xi' = \tfrac{1}{2}\Lambda_J + \tfrac{1}{2}\Lambda_K,$$

which expands to

$$\Xi' = e^{1234} + e^{1256} - \tfrac{1}{2}e^{1357} - \tfrac{1}{2}e^{1368} - \tfrac{1}{2}e^{1458} + \tfrac{1}{2}e^{1467}. \tag{16}$$

The Ξ'-Grassmannian consists of planes which are special Lagrangian with respect to both J and K. It is not hard to show that the Ξ'-and Ξ-geometries agree.

Indeed, it is enough to show that if ξ is special Lagrangian with respect to J, then ξ is special Lagrangian with respect to K if and only if it is complex with respect to I. Using the fact that for any complex structure, the Kähler calibration $\frac{1}{2}\omega^2$ is identically zero on the special Lagrangian Grassmannian $G(\Lambda)$, and the first identity in (14), it follows that Λ_I and $\frac{1}{2}\omega_K^2$ agree on $G(\Lambda_J)$. Therefore if a plane ξ in $G(\Lambda_J)$ is also in $G(\Lambda_K)$ then $\frac{1}{2}\omega_K^2(\xi) = 0$, whence $\Lambda_I(\xi) = 0$ so that $\xi \in G(\frac{1}{2}\omega_I^2)$. Similarly if ξ is in $G(\frac{1}{2}\omega_I^2)$, then $\Lambda_I(\xi) = 0$ whence $\frac{1}{2}\omega_K^2(\xi) = 0$ and $\xi \in G(\Lambda_K)$.

A useful convex linear combination of calibrations is the quaternionic calibration. Given a quaternionic structure $\{I, J, K\}$, we can define a quaternionic 4-form

$$\Theta_{\{I,J,K\}} \equiv \tfrac{1}{6}(\omega_I^2 + \omega_J^2 + \omega_K^2).$$

Being a convex linear combination of Kähler calibrations, $\Theta_{\{I,J,K\}}$ is also a calibration whose Grassmannian consists of planes which are complex with respect to each of the

complex structures I, J and K. For this reason e_{1234}, being a special Lagrangian plane relative to J cannot be quaternionic relative to Θ. We remedy this by defining another complex structure

$$J' = \begin{pmatrix} 1 & 2 & 5 & 6 \\ 3 & -4 & 7 & -8 \end{pmatrix},$$

which also anticommutes with I. Therefore $\{I' = I, J', K' = I'J'\}$ defines a quaternionic structure, with quaternionic form $\Theta \equiv \Theta_{\{I';,J';,K';\}}$ given by

$$\Theta = e^{1234} + \tfrac{1}{3}e^{1256} + \tfrac{1}{3}e^{1278} + \tfrac{1}{3}e^{1357} - \tfrac{1}{3}e^{1368} + \tfrac{1}{3}e^{1458} + \tfrac{1}{3}e^{1467}, \tag{17}$$

which now clearly calibrates e_{1234}. As in the case of the Kähler calibration, Θ is actually invariant under $\mathrm{Sp}_1 \cdot \mathrm{Sp}_2$; but because the Sp_1 factor is generated by the quaternionic structure itself, the quaternionic Grassmannian is actually the Sp_2-orbit of e_{1234}.

In contrast to a quaternionic structure, which consists of two anticommuting complex structures, let us consider two *commuting* complex structures: I defined in (7) and I'' defined by

$$I'' = \begin{pmatrix} 1 & 3 & 5 & 7 \\ 2 & -4 & -6 & 8 \end{pmatrix}. \tag{18}$$

Let us consider the self-dual form (again correcting a typo in [16])

$$\Sigma \equiv \tfrac{1}{4}\omega_I^2 - \tfrac{1}{4}\omega_{I''}^2 = e^{1234} + e^{1256} = (e_{12} + e_{78}) \wedge (e_{34} + e_{56}). \tag{19}$$

In order to see that this form is a calibration, it is easiest to rewrite it as a convex linear combination of special Lagrangian forms

$$\Sigma = \tfrac{1}{2}\Lambda_J + \tfrac{1}{2}\Lambda_{J''},$$

where J is the complex structure in (9) and J'' is given by

$$J'' = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -8 & -7 & 6 & 5 \end{pmatrix},$$

which corrects yet another typo in [16]. The complex structures J and J'' are also commuting. Moreover, I'' and J'' anticommute, whence $\{I, J, K = IJ\}$ and $\{I'', J'', K'' = I''J''\}$ are two commuting quaternionic structures.

From the product form of Σ in (19), we see that the Σ-planes are products of a complex plane in the span of e_{1278} relative to $I_1 = \begin{pmatrix} 1 & 7 \\ 2 & 8 \end{pmatrix}$ and a complex plane relative to $I_2 = \begin{pmatrix} 3 & 5 \\ 4 & 6 \end{pmatrix}$ in the span of e_{3456}. Equivalently,[2] Σ-planes are products of a special Lagrangian

[2] This exemplifies the fact that 2-planes in $\mathbb{E}^4$ which are complex relative to a Kähler calibration ω_I, are special Lagrangian relative to Λ_J, where I, J, and $K = IJ$ defines a quaternionic structure. This is because of the isomorphism $\mathrm{SU}_2 \cong \mathrm{Sp}_1$, so that the SU_2 which leaves I invariant actually leaves invariant a quaternionic structure.

plane relative to $J_1 = \begin{pmatrix} 1 & 2 \\ 8 & 7 \end{pmatrix}$ in the span of e_{1278} and a special Lagrangian plane relative to $J_2 = \begin{pmatrix} 3 & 4 \\ 6 & 5 \end{pmatrix}$ in the span of e_{3456}. Σ is invariant under an $\mathrm{U}_2 \times \mathrm{U}_2$ subgroup of SO_8. The centre stabilises e_{1234} — in fact, it stabilises the spans of e_{12} and e_{34} separately — whence the Σ-Grassmannian is the $\mathrm{Sp}_1 \times \mathrm{Sp}_1$ orbit of e_{1234}.

Finally we point out that the calibrations corresponding to the $(3, i)$ geometries can be constructed out of complex and quaternionic structures. In fact, we have the following expressions

$$\Psi_{(3,1)} = \tfrac{3}{4}\Theta + \tfrac{1}{4}\Omega_J, \qquad \Psi_{(3,2)} = \tfrac{3}{5}\Theta - \tfrac{2}{5}\omega_{I''}^2, \qquad \Psi_{(3,3)} = \tfrac{1}{2}\Theta - \tfrac{1}{2}\tilde{\Theta},$$

where J and I'' are the complex structures defined by (9) and (18), respectively, and where $\tilde{\Theta} \equiv \Theta_{\{\tilde{I},\tilde{J},\tilde{K}\}}$, given by

$$\hat{\Theta} = -e^{1234} - \tfrac{1}{3}e^{1256} + \tfrac{1}{3}e^{1278} + \tfrac{1}{3}e^{1357} + \tfrac{1}{3}e^{1368} - \tfrac{1}{3}e^{1458} + \tfrac{1}{3}e^{1467},$$

is the quaternionic calibration corresponding to the quaternionic structure generated by $\tilde{I} = I''$ in (18) and

$$\tilde{J} = \begin{pmatrix} 1 & 2 & 5 & 6 \\ 3 & 4 & 7 & 8 \end{pmatrix}.$$

6. Generalised self-duality

Every (constant coefficient) 4-form φ in $\mathbb{E}^8$ defines an endomorphism of the space of 2-forms:

$$\hat{\varphi} : \bigwedge^2 \mathbb{E}^8 \to \bigwedge^2 \mathbb{E}^8, \qquad \omega \mapsto \star(\star\varphi \wedge \omega). \tag{20}$$

Explicitly, if $\varphi = \sum_{i<j<k<l} \varphi_{ijkl} e_{ijkl}$ and $\omega = \sum_{i<j} \omega_{ij} e_{ij}$ then

$$(\hat{\varphi}\omega)_{ij} = \sum_{k<l} \varphi_{ijkl}\, \omega_{kl}.$$

This expression clearly shows that $\hat{\varphi}$ is traceless, and symmetric under the natural inner product

$$\langle \alpha, \beta \rangle \equiv \star(\alpha \wedge \star\beta) = \sum_{i<j} \alpha_{ij}\, \beta_{ij}$$

on the space of 2-forms. This means that $\hat{\varphi}$ will be diagonalisable. If G is the SO_8-isotropy subgroup of φ, then the eigenspaces of $\hat{\varphi}$ are G-submodules of $\bigwedge^2 \mathbb{E}^8$, the $\underline{28}$ or adjoint representation $\mathfrak{so}_8$ of SO_8. A canonical G-submodule is the adjoint representation $\mathfrak{g} \subset \mathfrak{so}_8$, but of course there are other G-submodules as well. One can use $\hat{\varphi}$ to define a generalised *self-duality* for 2-forms in eight dimensions by demanding that a 2-form belongs to a definite

G-submodule of $\mathfrak{so}_8$. This generalises self-duality in four dimensions, where we can take $\varphi = \star 1$, and $\hat{\varphi} = \star$ itself. The eigenspaces of $\hat{\varphi}$ in this case are the subspaces of self-dual and anti-self-dual 2-forms: corresponding to the adjoint representations of the two Sp_1 factors in $\mathrm{SO}_4 \cong \mathrm{Sp}_1 \cdot \mathrm{Sp}_1$.

Generalised self-duality plays a crucial role in the attempts to generalise the notion of Yang–Mills instantons to higher dimensions [13,44]. Suppose that the Yang–Mills curvature $F(A)$ satisfies a generalised self-duality condition

$$\hat{\varphi}\, F(A) = c\, F(A) \tag{21}$$

for some *non-zero* constant c. Then one easily computes

$$d_A \star F(A) = c^{-1}\, \mathrm{d}_A(\star\varphi \wedge F(A)) = c^{-1}(\mathrm{d} \star \varphi \wedge F(A) + \star\varphi \wedge \mathrm{d}_A\, F(A)),$$

whence using the Bianchi identity $d_A F(A) = 0$ and *provided that* $\star\varphi$ *is closed*, the Yang–Mills equations of motion are satisfied automatically. In the geometries under consideration φ is self-dual and it is constant, so that it is co-closed.

In what follows we will discuss the possible notions of self-duality which are available for each of the above geometries in eight dimensions, by analysing the eigenspace decompositions of the endomorphisms $\hat{\varphi}$ corresponding to the different calibrations φ described above. We should remark however that despite the fact that a one-to-one correspondence between geometries and generalised self-duality conditions is not expected — after all self-duality depends crucially on the calibration, whereas as we saw above for the case of the complex Lagrangian geometry, different calibrations can give rise to the same geometry — nevertheless we will see that in some cases the geometry does determine the possible generalised self-dualities.

6.1. Cayley geometry

The Cayley calibration (12) is invariant under a $\mathfrak{spin}_7$-subgroup of SO_8, under which the $\underline{28}$ breaks up as

$$\underline{28} \to \underline{7} \oplus \underline{21},$$

where the $\underline{21}$ corresponds to the adjoint representation $\mathfrak{spin}_7 \subset \mathfrak{so}_8$. It is well known that the endomorphism $\hat{\Omega}$ obeys the following characteristic polynomial:

$$(\hat{\Omega} - \mathbb{1})(\hat{\Omega} + 3\mathbb{1}) = 0,$$

whence we see that the eigenvalues are 1 and -3, and (using tracelessness of $\hat{\Omega}$) with multiplicities 21 and 7, respectively. Therefore there are two possible notions of self-duality, and hence two possible extensions of the notion of instanton to eight dimensions. As shown in [1], supersymmetry seems to prefer the definition of instanton which says that $F(A)$ belongs to $\mathfrak{spin}_7 \subset \mathfrak{so}_8$: $\hat{\Omega} F(A) = F(A)$. Gauge fields satisfying this relation are known as *octonionic instantons*, for reasons explained in [19].

6.2. Complex geometries

Let I denote the complex structure defined in Eq. (7), and let $\Upsilon \equiv \frac{1}{2}\omega_I^2$, which is given by (8). Let $\hat{\Upsilon}$ denote the endomorphism of 2-forms defined from Υ according to (20). Its characteristic polynomial is given by

$$(\hat{\Upsilon} - 3\mathbb{1})(\hat{\Upsilon} - \mathbb{1})(\hat{\Upsilon} + \mathbb{1}) = 0,$$

whence Υ has three eigenvalues 3, 1 and -1. The multiplicities are 1, 12 and 15, respectively. Υ is U_4 invariant, and under $\mathrm{U}_4 \cong (\mathrm{SU}_4 \times \mathrm{U}_1)/\mathbb{Z}_4$ the $\underline{28}$ breaks up as

$$\underline{28} \to \underline{6}_2 \oplus \underline{6}_{-2} \oplus \underline{15}_0 \oplus \underline{1}_0,$$

where the last two factors correspond to the adjoint representation $\mathfrak{u}_4 = \mathfrak{su}_4 \oplus \mathfrak{u}_1 \subset \mathfrak{so}_8$, and where the first two factors together make up an irreducible *real* representation of dimension 12. Therefore there is no accidental degeneracy in the eigenspace decomposition of $\hat{\Upsilon}$, in the sense that the group theory does not refine any further the eigenvalues of $\hat{\Upsilon}$. The natural self-duality condition in gauge theory is the one which says that $F(A)$ belongs to $\mathfrak{su}_4 \subset \mathfrak{so}_8$: $\hat{\Upsilon} F(A) = -F(A)$. These equations are the well-known Kähler–Yang–Mills equations, studied in [17,43].

6.3. Special Lagrangian geometry

Let $\Lambda \equiv \Lambda_J$ denote the special Lagrangian form defined by Eq. (11). It is invariant under $\mathrm{SU}_4 \subset \mathrm{SO}_8$, under which the $\underline{28}$ breaks up as

$$\underline{28} \to 2\,\underline{6} \oplus \underline{1} \oplus \underline{15},$$

where now each $\underline{6}$ is a real representation of $\mathrm{SU}_4 \cong \mathrm{Spin}_6$. The map $\hat{\Lambda}$ on 2-forms obeys the following characteristic polynomial:

$$(\hat{\Lambda} + 2\mathbb{1})\hat{\Lambda}(\hat{\Lambda} - 2\mathbb{1}) = 0,$$

whence it has eigenvalues -2, 0 and 2. The multiplicities can easily worked to be 6, 16 and 6, which shows that the eigenvalue 0 is degenerate.

6.4. Complex Lagrangian geometry

Let Ξ denote the complex Lagrangian calibration given by (15). The corresponding endomorphism $\hat{\Xi}$ satisfies the characteristic polynomial

$$\left(\hat{\Xi} - \tfrac{5}{2}\mathbb{1}\right)\left(\hat{\Xi} - \tfrac{3}{2}\mathbb{1}\right)\left(\hat{\Xi} - \tfrac{1}{2}1\mathbb{1}\right)\left(\hat{\Xi} + \tfrac{1}{2}\mathbb{1}\right)\left(\hat{\Xi} + \tfrac{3}{2}\mathbb{1}\right) = 0,$$

so that it has five eigenvalues: $\frac{5}{2}$, $\frac{3}{2}$, $\frac{1}{2}$, $-\frac{1}{2}$ and $-\frac{3}{2}$. The multiplicities are 1, 5, 6, 11 and 5, which again agrees with $\hat{\Xi}$ being traceless. The eigenvalues $\pm\frac{1}{2}$ are now degenerate,

a fact for which there seems to be no group-theoretical explanation, since Ξ is precisely Sp_2-invariant, and under Sp_2 the $\underline{28}$ breaks up as

$$\underline{28} \to 3\,\underline{1} \oplus 3\,\underline{5} \oplus \underline{10}.$$

The three singlets correspond to ω_I, ω_J and ω_K, and the $\underline{10}$ corresponds to the adjoint representation $\mathrm{sp}_2 \subset \mathfrak{so}_8$.

Similarly, let Ξ' be the other complex Lagrangian calibration defined by (16) and let $\hat{\Xi}'$ be the corresponding endomorphism. It satisfies the characteristic polynomial

$$(\hat{\Xi}' + 2\mathbb{1})(\hat{\Xi}' + \mathbb{1})\hat{\Xi}'(\hat{\Xi}' - \mathbb{1})(\hat{\Xi}' - 2\mathbb{1}) = 0.$$

The five eigenvalues -2, -1, 0, 1 and 2 have multiplicities 5, 2, 10, 10 and 1, respectively. $\hat{\Xi}'$ is actually invariant under an $\mathrm{U}_1 \cdot \mathrm{Sp}_2 = (\mathrm{U}_1 \times \mathrm{Sp}_2)/\mathbb{Z}_2$ subgroup of SO_8. Under this subgroup the $\underline{28}$ breaks up as

$$\underline{28} \to \underline{1}_0 \oplus \underline{10}_0 \oplus \underline{5}_0 \oplus \underline{1}_2 \oplus \underline{1}_{-2} \oplus \underline{5}_2 \oplus \underline{5}_{-2}. \tag{22}$$

The first two factors correspond to the adjoint representation $\mathfrak{u}_1 \oplus \mathrm{sp}_2 \subset \mathfrak{so}_8$. In terms of real representations, $\underline{1}_2 \oplus \underline{1}_{-2}$ is an irreducible 2-dimensional representation and $\underline{5}_2 \oplus \underline{5}_{-2}$ is an irreducible 10-dimensional representation. Therefore there is no degeneracy in the spectrum of $\hat{\Xi}'$.

6.5. *Quaternionic geometry*

The linear map $\hat{\Theta}$ associated to the quaternionic 4-form Θ in (17) obeys the characteristic polynomial

$$(\hat{\Theta} + 1\mathbb{1})\left(\hat{\Theta} - \tfrac{1}{3}1\mathbb{1}\right)\left(\hat{\Theta} - \tfrac{5}{3}\mathbb{1}\right) = 0,$$

so that it has three eigenvalues: -1, $\frac{1}{3}$ and $\frac{5}{3}$. The multiplicities are 10, 15 and 3, respectively, reiterating the fact that $\hat{\Theta}$ is traceless. The degeneracy of the eigenvalues is easily explained if we remark that Θ is actually invariant under the maximal subgroup $\mathrm{Sp}_1 \cdot \mathrm{Sp}_2 = (\mathrm{Sp}_1 \times \mathrm{Sp}_2)/\mathbb{Z}_2$ of SO_8, under which the $\underline{28}$ decomposes into three factors as

$$\underline{28} \to (\underline{3}, \underline{1}) \oplus (\underline{1}, \underline{10}) \oplus (\underline{3}, \underline{5}),$$

the first two factors corresponding to the adjoint representation $\mathrm{sp}_1 \oplus \mathrm{sp}_2 \subset \mathfrak{so}_8$. The corresponding self-duality equations for Yang–Mills fields were originally studied, in the context of quaternionic Kähler manifolds, in [20,32].

6.6. *Sub-quaternionic geometries*

Finally let us consider the self-dual forms defined by (5). Let $\hat{\Psi}_i$ denote the endomorphism of 2-forms defined by the calibration $\Psi_{(3,i)}$. These maps obey the following characteristic polynomials:

Table 2
Geometries omitted from Table 1

$(p\|n)$	Group G	Isotropy H	Geometry G/H	Fraction ν
(5|10)	$U_1 \times SU_2$	?	?	$\frac{3}{32}$
(5|10)	U_1	{1}	$G(1\|2)$?	$\frac{1}{8}$

$$(\hat\Psi_1 - \tfrac{3}{2}\mathbb{1})(\hat\Psi_1 - \tfrac{1}{2}\mathbb{1})(\hat\Psi_1 + \tfrac{1}{2}\mathbb{1})(\hat\Psi_1 + \tfrac{3}{2}\mathbb{1}) = 0,$$
$$(\hat\Psi_2 - \tfrac{7}{5}\mathbb{1})(\hat\Psi_2 - \tfrac{3}{5}\mathbb{1})(\hat\Psi_2 + \tfrac{1}{5}\mathbb{1})(\hat\Psi_2 + \mathbb{1})(\hat\Psi_2 + \tfrac{9}{5}\mathbb{1}) = 0,$$
$$(\hat\Psi_3 - \tfrac{4}{3}\mathbb{1})(\hat\Psi_3 - \tfrac{2}{3}\mathbb{1})\hat\Psi_3(\hat\Psi_3 + \tfrac{2}{3}\mathbb{1})(\hat\Psi_3 + \tfrac{4}{3}\mathbb{1}) = 0.$$

The multiplicities of the eigenvalues are given as follows: for $\hat\Psi_1$ the eigenvalues are $\frac{3}{2}$, $\frac{1}{2}$, $-\frac{1}{2}$, and $-\frac{3}{2}$ with multiplicities 3, 12, 9 and 4, respectively; for $\hat\Psi_2$ the eigenvalues are $\frac{7}{5}$, $\frac{3}{5}$, $-\frac{1}{5}$, -1 and $-\frac{9}{5}$ and $-\frac{3}{2}$ with multiplicities 3, 9, 9, 6 and 1, respectively; and for $\hat\Psi_3$ the eigenvalues are $\frac{4}{3}$, $\frac{2}{3}$, 0, $-\frac{2}{3}$ and $-\frac{4}{3}$ with multiplicities 3, 6, 10, 6 and 3, respectively. The forms $\Psi_{(3,i)}$ are invariant with respect to the subgroups $K_{(3,i)}$ of SO_8 given in Eq. (6). As mentioned above, these groups are subgroups of the $Sp_1 \cdot Sp_2$ isotropy of the quaternionic form Θ in Eq. (17). In fact, the first Sp_1 factor in $K_{(3,i)}$ is precisely the same as the one in $Sp_1 \cdot Sp_2$. All eigenspace decompositions are degenerate for these three groups. As an example, let us work out the (3, 1) geometry. Under $K_{(3,1)}$ the $\underline{28}$ breaks up as

$$\underline{28} \to (\underline{1}, \underline{1}, \underline{3}) \oplus (\underline{1}, \underline{3}, \underline{1}) \oplus 2(\underline{3}, \underline{1}, \underline{1}) \oplus (\underline{1}, \underline{2}, \underline{2}) \oplus (\underline{3}, \underline{2}, \underline{2}),$$

which shows that the $-\frac{1}{2}$ eigenvalue is degenerate. Similar considerations hold for the (3, 2) and (3, 3) geometries. The generalised self-dual Yang–Mills equations have not been studied for these geometries. They may provide an interesting refinement to the self-dual Yang–Mills equations in quaternionic Kähler geometry.

7. Conclusion

In this paper we have presented a survey of some of the calibrated geometries which have occurred in recent studies on the local singularity structure of supersymmetric fivebranes in M-theory [2–4]. Some of these geometries appeared explicitly in [22,23] and implicitly in some earlier work [8,37,40,42]. Calibrated geometries have also appeared in related contexts in other papers [5,6,9,24,38,41]. Calibrated geometry is therefore beginning to emerge as the natural language in which to phrase geometric questions in the study of branes. An appropriate slogan might be *brane geometry is calibrated geometry*.

Not all geometries which have appeared in our work have been showcased here. Our choice reflects the present level of knowledge in this topic. We have omitted two of the subgeometries of $G(5|10)$ which were obtained in [2], because we were not able to identify them. They are summarised in Table 2 below. The systematic study of the faces of $G(p|n)$ has alas stopped short of the interesting $G(5|10)$ case: only partial results are known for

$G(4|8)$ and very little indeed for $n > 8$. It is hoped that this survey might help to rekindle the interest in this problem.

Finally, it should be mentioned that the calibrated subgeometries of the Grassmannians $G(p|n)$ are far richer than what has been surveyed in this paper. We have only looked at geometries defined by *supersymmetric* brane configurations; whereas other calibrated geometries describe non-supersymmetric configurations whose study might still be physically interesting, since they correspond to local singularities of minimal submanifolds, which presumably still give rise to stable states.

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References

[1] B.S. Acharya, J.M. Figueroa-O'Farrill, M.O'Loughlin, B. Spence, Euclidean D-branes and higher-dimensional gauge theory, Nucl. Phys. B 514 (1998) 583–602, hep-th/9707118.
[2] B.S. Acharya, J.M. Figueroa-O'Farrill, B. Spence, Planes, branes and automorphisms: I. Static branes, hep-th/9805073.
[3] B.S. Acharya, J.M. Figueroa-O'Farrill, B. Spence, Branes at angles and calibrated geometry, JHEP 4 (1998) 12, hep-th/9803260.
[4] B.S. Acharya, J.M. Figueroa-O'Farrill, B. Spence, S. Stanciu, Planes, branes and automorphisms: II. Branes in motion, hep-th/9805176.
[5] K. Becker, M. Becker, D.R. Morrison, H. Ooguri, Y. Oz, Z. Yin, Supersymmetric cycles in exceptional holonomy manifolds and Calabi–Yau 4-folds, Nucl. Phys. 480 (1996) 225–238, hep-th/9608116.
[6] K. Becker, M. Becker, A. Strominger, Fivebranes, membranes and non-perturbative string theory, Nucl. Phys. 456 (1995) 130–152, hep-th/9509175.
[7] M. Berger, Du côtè de chez Pu, Ann. Sci. École Norm. Sup. 5 (1972) 1–44.
[8] M. Berkooz, M.R. Douglas, R.G. Leigh, Branes intersecting at angles, Nucl. Phys. B 480 (1996) 265–278, hep-th/9606139.
[9] M. Bershadsky, V. Sadov, C. Vafa, D-branes and topological field theory, Nucl. Phys. B 463 (1996) 420–434, hep-th/9511222.
[10] A.L. Besse, Einstein Manifolds, Springer, Berlin, 1987.
[11] R.L. Bryant, Remarks on spinors in low dimension, Unpublished notes, 1998.
[12] R.L. Bryant, Metrics with exceptional holonomy, Ann. Math. 126 (1987) 525–576.
[13] E. Corrigan, C. Devchand, D.B. Fairlie, J. Nuyts, First-order equations for gauge fields in spaces of dimension greater than four, Nucl. Phys. B 214 (1983) 452–464.
[14] J. Dadok, F.R. Harvey, Calibrations on $\mathbb{R}^6$, Duke Math. J. 50 (1983) 1231–1243.
[15] J. Dadok, F.R. Harvey, Calibrations and spinors, Acta Math. 170 (1993) 83–120.
[16] J. Dadok, F.R. Harvey, F. Morgan, Calibrations on $\mathbb{R}^8$, Trans. Am. Math. Soc. 307 (1988) 1–40.
[17] S.K. Donaldson, Anti-self-dual Yang–Mills connections on complex algebraic surfaces and stable vector bundles, Proc. London Math. Soc. 3 (1985) 1–26.
[18] H. Federer, Geometric Measure Theory, Springer, New York, 1959.
[19] J.M. Figueroa-O'Farrill, Gauge theory and the division algebras, J. Geom. Phys. 32 (1999) 227–240, hep-th/9710168.

[20] K. Galicki, Y.S. Poon, Duality and Yang–Mills fields on quaternionic Kähler manifolds, J. Math. Phys. 32 (1991) 1263–1268.
[21] J.P. Gauntlett, Intersecting branes, hep-th/9705011.
[22] J.P. Gauntlett, N.D. Lambert, P.C. West, Branes and calibrated geometries, hep-th/9803216.
[23] G.W. Gibbons, G. Papadopoulos, Calibrations and intersecting branes, hep-th/9803163.
[24] M. Grabowski, C.H. Tze, Generalized self-dual bosonic membranes, vector cross-products and analyticity in higher dimensions, Phys. Lett. B 224 (1989) 259–264.
[25] F.R. Harvey, Spinors and Calibrations, Academic Press, New York, 1990.
[26] F.R. Harvey, H.B. Lawson, Calibrated geometries, Acta Math. 148 (1982) 47–157.
[27] F.R. Harvey, F. Morgan, The comass ball in $\bigwedge^3(\mathbb{R}^6)^*$, Ind. U. Math. J. 35 (1986) 145–156.
[28] F.R. Harvey, F. Morgan, The faces of the Grassmannian of three-planes in $\mathbb{R}^7$ (Calibrated geometries in $\mathbb{R}^7$), Invent. Math. 83 (1986) 191–228.
[29] G. Lawlor, The angle criterion, Invent. Math. 95 (1989) 437–446.
[30] G. Lawlor, Area-minimizing m-tuples of k-planes, in: Th. M. Rassias (Ed.), The Problem of Plateau, World Scientific, Singapore, 1992, pp. 165–180.
[31] H.B. Lawson, M.L. Michelsohn, Spin Geometry, Princeton University Press, Priceton, NJ, 1989.
[32] M. Mamone-Capria, S.M. Salamon, Yang–Mills fields on quaternionic spaces, Nonlinearity 1 (1988) 517–530.
[33] F. Morgan, On the singular structure of three-dimensional, area-minimizing surfaces, Trans. Am. Math. Soc. 276 (1983) 137–143.
[34] F. Morgan, The exterior algebra $\bigwedge^k \mathbb{R}^n$ and area minimization, Linear Algebra Appl. 66 (1985) 1–28.
[35] F. Morgan, Area-minimizing surfaces, faces of Grassmannians, and calibrations, Am. Math. Monthly 95 (1988) 813–822.
[36] D. Nance, Sufficient condition for a pair of n-planes to be area-minimizing, Math. Ann. 279 (1987) 161–164.
[37] N. Ohta, P.K. Townsend, Supersymmetry of M-branes at angles, Phys. Lett. B 418 (1998) 77, hep-th/9710129.
[38] H. Ooguri, Y. Oz, Z. Yin, D-branes on Calabi–Yau spaces and their mirrors, Nucl. Phys. B 477 (1996) 407–430.
[39] R. Penrose, W. Rindler, Spinors and Space–Time, Cambridge University Press, Cambridge, 1984.
[40] M.M. Sheikh Jabbari, Classification of different branes at angles, hep-th/9710121.
[41] S. Stanciu, D-branes in Kazama–Suzuki models, hep-th/9708166.
[42] P.K. Townsend, M-branes at angles, hep-th/9708074.
[43] K. Uhlenbeck, S.T. Yau, On the existence of Hermitian Yang–Mills connections on stable bundles over compact Kähler manifolds, Commun. Pure Appl. Math. 39 (1986) 257–293; Correction: Commun. Pure Appl. Math. 42 (1986) 703–707.
[44] R.S. Ward, Completely solvable gauge-field equations in dimension greater than four, Nucl. Phys. B 236 (1984) 381–396.